Diffusion by extrinsic noise in the kicked Harper map

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A significantly improved analytic understanding of the extrinsically driven diffusion process is presented in a nonlinear dynamical system in which the phase space is divided into periodic two-dimensional tiles of regular motion, separated by a connected separatrix network (web) [previously studied by A. J. Lichtenberg and Blake P. Wood, Phys. Rev. Lett. **62**, 2213 (1989)]. The system is represented by the usual ''kicked Harper map'' with added extrinsic noise terms. Three different diffusion regimes are found depending upon the strength of the extrinsic perturbation l relative to the web and regular motions. When the extrinsic noise is dominant over the intrinsic stochasticity and the regular rotation motions in the tile, diffusion obeys the random phase scaling l^2. When the extrinsic noise is dominant over the intrinsic stochasticity, but weaker than the regular rotation motion, the diffusion scales as $lK^{1/2}$, where K is the strength of the intrinsic kick. These findings agree well with numerical simulation results. When the extrinsic noise process is weaker than the stochastic web process, we analytically reproduce the well-known numerical result: The web diffusion is reduced by the ratio of phase-space areas of intrinsic to extrinsic stochasticity.

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I. INTRODUCTION

An important problem in nonlinear dynamics is understanding the effect of extrinsic stochasticity on the diffusion across a divided phase space. In two degrees of freedom the divided phase space can consist of regions dominated by KAM (Kol'mogorov-Arnol'd-Moser) curves and regions connected by intrinsic stochasticity [1]. If the system is above the stochasticity threshold and primarily connected with intrinsic stochasticity regions (with embedded KAM islands), the extrinsic stochasticity can slow down the intrinsic diffusion by moving the phase points between the intrinsic stochastic region and the KAM islands [2]. On the other hand, if the intrinsic stochasticity regions are isolated from each other by phase-spanning KAM regions, the global diffusion rate is determined by the slow extrinsic steps across the KAM curves [2–4]. In the actual physical systems, the latter situation is often preferred since the former produces diffusion which is uncontrollably too fast.

A more interesting problem with regard to the extrinsic diffusion rises in the case when the phase space is divided into periodic two-dimensional tiles, which appears commonly when a linear oscillator is resonantly perturbed. The connected boundary region between the tiles (separatrix) can easily be stochastic when the intrinsic perturbation is reasonably large, to form a stochastic connected web, which leads to rapid intrinsic diffusion over large scale. The KAM regions in this case are locally confined, isolated from each other by the connected stochastic web. Thus, in the absence of extrinsic noise, there is a fast global diffusion by the large scale intrinsic web diffusion. The role of extrinsic noise in this case can be quite different. When the web diffusion is strong, the extrinsic noise reduces the global diffusion rate as usual. The difference stands out in the case when the stochastic web diffusion is weak: Within each tile, the phase points rotate rapidly along the KAM curves of Fig. 1. Extrinsic noise can scatter the rotating phase points between the tiles across the tile boundaries. The diffusive step size is then equal to the tile size, which is much greater than the normal extrinsic step size across each KAM curves.

This problem was studied numerically in Ref. [5] using an infinitely periodic web-tile structured map in two dimen-

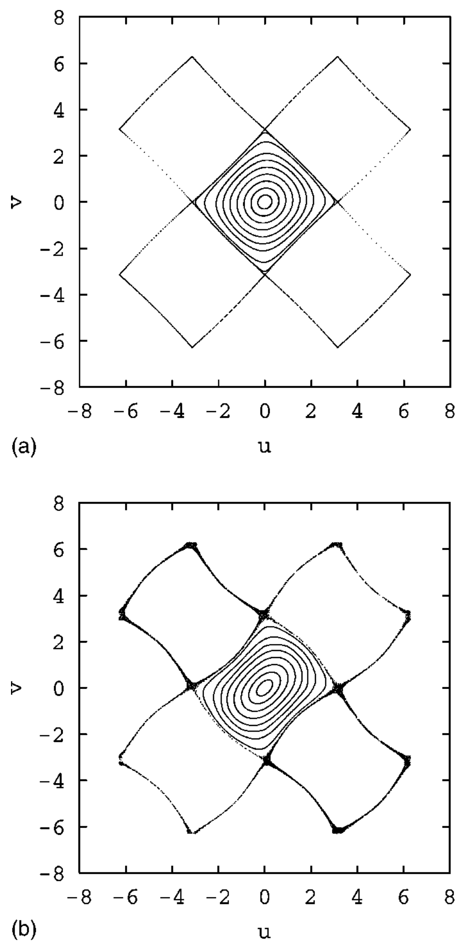

FIG. 1. Phase-space plots of Eq. (1) without external noise. (a) is for $K=0.1$ and (b) is for $K=0.5$.

sions. They reported the findings that when the intrinsic web stochasticity dominates over the extrinsic stochasticity, the global web diffusion rate is reduced by ‘‘the ratio of phase-space areas in the intrinsic to extrinsic stochasticity,’’ and that, when the extrinsic stochasticity dominates over the web stochasticity the global diffusion rate shows a new trend set by the extrinsic noise. For the latter case, they have offered an analytic explanation ($\propto l$, where l is the extrinsic noise strength) of the diffusion behavior without being able to explain the K dependence whose existence was obvious from their numerical result.

It is the purpose of the present work to offer an analytic explanation of the extrinsically driven diffusion behavior in the same system as in Ref. [5], where the phase space has an infinitely periodic homogeneous web-tile structure. After a minor, but clear, explanation of the reduction in the global web diffusion rate by the ratio of phase-space areas when the diffusion is dominated by the intrinsic stochasticity, we show that the dominance of the extrinsic stochasticity appears differently in two different regimes separated by the relative strength of the extrinsic noise frequency to the intrinsic regular rotation frequency. New diffusion scalings have been identified analytically when the extrinsic stochasticity dominates over the intrinsic stochasticity, which agrees well with numerical simulation results.

We follow the procedure of Ref. [5] and start with the mapping representation of the kicked oscillator [6,7]

$$v_{n+1}=-(u_n+K\sin v_n)\sin\alpha+v_n\cos\alpha,$$

$$u_{n+1}=(u_n+K\sin v_n)\cos\alpha+v_n\sin\alpha,$$

where u and v are the two-dimensional quantities oscillating at the angular frequency ω, $\alpha=\omega T$ is the rotation angle of the oscillator between kicks, T is the time interval between kicks, and K is the kick amplitude. At a resonance we have $\alpha=2\pi p/q$. Taking $p=1$ and $q=4$ (four kicks per oscillation), iterating four times, and keeping only the lowest-order terms in K, we obtain the so-called ‘‘kicked Harper map’’ [8]

$$\begin{aligned} v_{n+1}&=v_n-2K\sin u_n+u_r,\\ u_{n+1}&=u_n+2K\sin v_{n+1}+v_r, \end{aligned} \tag{1}$$

where the extrinsic noise terms u_r and v_r are added. In the present work we use uniformly distributed random variables between $\pm l$ for u_r and v_r. Without the noise terms, the mapping is area preserving.

In Fig. 1 we display the phase-space plots for a tile without extrinsic noise ($l=0$) for $K=0.1$ and 0.5 showing the stochastic web region and regular KAM region. Figure 1(a) is the case with small web diffusion and Fig. 1(b) is the case where the web diffusion D_{web} begins to be significant. As the K value is increased further, distortion of the KAM surfaces occur within the tile. The extrinsic noise introduces scattering across the KAM surfaces, giving connections between different KAM surfaces, between the KAM region and the web region, and between different tiles. We first present a minor analytic discussion in the case when the intrinsic web stochasticity dominates over the extrinsic stochasticity, followed by a main discussion in the regimes where the intrinsic web stochasticity is small.

II. EXTRINSIC NOISE EFFECT ON STOCHASTIC WEB DIFFUSION

In the case where the intrinsic web diffusion dominates over the extrinsic diffusion it is well known numerically that the role of extrinsic noise is to scatter the phase points between the rapidly diffusing global intrinsic stochasticity region and the local KAM region where the slower extrinsic diffusion is in action. As a result, the global web diffusion rate D_{web} is reduced. In order to understand the reduction amount, we model that the trajectory of a phase point is switched back and forth between the web and KAM region, and we neglect the slow extrinsic diffusion in the KAM region. During the period Δ of the phase point trapping in the KAM region, the diffusion process is turned off. Defining a measurable radial quantity $r=\sqrt{u^2+v^2}$, we write a modified Langevin equation

$$\frac{dr(t)}{dt}=\xi(t)\left[1-\sum_{i=1}^{N(t)}\Theta(t-t_i)\Theta(t_i+\Delta-t)\right]=\xi(t)H(t), \tag{2}$$

where t_i represents the instant at which the ith trapping event occurs, $\Theta(x)$ is the usual step function such that $\Theta(x)=1$ for $x\geqslant 0$, $\Theta(x)=0$ for $x<0$, $N(t)$ is a stochastic number function denoting the number of trapping events occurred up to time t, and $\xi(t)$ is the stochastic displacement function of the white-noise type inside the web region in such a way that

$$\langle\xi(t)\xi(t')\rangle_\xi=2D_{\mathrm{web}}\delta(t-t').$$

Here $\langle\cdots\rangle_\xi$ denotes the ensemble average over ξ. This form of the Langevin equation includes the phenomenon that when the time belongs in the KAM-confined period, $t_i\leqslant t<t_i+\Delta$, the radial velocity is zero.

From this Langevin equation we can compute directly the web diffusion rate modified by the extrinsic noise effect,

$$
\begin{aligned}
D^l_{\rm web} &= \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left\langle \int_t^{t+\Delta t}dt'\int_t^{t+\Delta t}dt''\xi(t')H(t')\xi(t'')H(t'')\right\rangle_{\xi,N(t)}\\
&= \lim_{\Delta t\to 0}\frac{1}{2\Delta t}\left\langle \int_t^{t+\Delta t}dt'\int_t^{t+\Delta t}dt'' H(t')H(t'')2D_{\rm web}\delta(t'-t'')\right\rangle_{N(t)}\\
&= D_{\rm web}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\langle H^2(t')\rangle_{N(t)}\\
&= D_{\rm web}\lim_{\Delta t\to 0}\frac{1}{\Delta t}\int_t^{t+\Delta t}dt'\langle H(t')\rangle_{N(t)}\\
&= D_{\rm web}\left\{1-\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left\langle \int_0^{t+\Delta t}dt' O(t')-\int_0^t dt' O(t')\right\rangle_{N(t)}\right\},
\end{aligned}
$$

where we have used the fact that $H(t')$ is either 1 or 0, and defined a stochastic function $O(t)$ as $\Sigma_{i=1}^{N(t)}\Theta(t-t_i)\Theta(t_i+\Delta-t)$.

The integration of $O(t)$ over t can be performed as follows:

$$
\int_0^t dt' O(t') = \begin{cases} \Delta N(t) & \text{if } t_{N(t)}+\Delta\leqslant t< t_{N(t)+1}\\ \Delta(N(t)-1)+t-t_{N(t)} & \text{if } t_{N(t)}\leqslant t< t_{N(t)}+\Delta. \end{cases}
$$

Then in the asymptotic time limit $N(t)\gg 1$, this integration simply becomes $\Delta N(t)$, yielding

$$
\begin{aligned}
D^l_{\rm web} &= D_{\rm web}\left\{1-\lim_{\Delta t\to 0}\Delta\frac{\langle N(t+\Delta t)-N(t)\rangle}{\Delta t}\right\}\\
&= D_{\rm web}\left\{1-\Delta\frac{d\langle N(t)\rangle}{dt}\right\}. \qquad (3)
\end{aligned}
$$

Defining ν as the average number of trapping events per unit time, we can identify $\langle N(t)\rangle$ as νt. Assuming that the average trapping time Δ is much shorter than the intertrapping time $1/\nu$ in the present case where the intrinsic web diffusion is dominant, we have

$$
D^l_{\rm web} = D_{\rm web}\{1-\nu\Delta\}\simeq\frac{D_{\rm web}}{1+\nu\Delta}\quad \text{for } \nu\Delta\ll 1. \qquad (4)
$$

Using the equilibrium hypothesis that the phase space is uniformly populated by the diffusing phase points, we relate the steady-state probability P_W (P_T) for a phase point to belong to the web (KAM tile) and the phase-space area A_W (A_T) of the web region (tile region) by the equation $P_W/P_T = A_W/A_T$. Adopting the detailed balancing principle satisfied in the equilibrium situation, i.e., $P_W P_{W\to T} = P_T P_{T\to W}$, where $P_{W\to T}$ represents the transition probability from the web to the inner tile region and vice versa, we have $P_{W\to T}/P_{T\to W} = \nu\Delta = P_T/P_W = A_T/A_W$. Using this result in Eq. (4) we have

$$
D^l_{\rm web}\simeq\frac{D_{\rm web}}{1+A_T/A_W} = \frac{A_W}{A}D_{\rm web}\quad \text{for } A_T/A_W\ll 1, \qquad (5)
$$

where A denotes the total phase-space area $A = A_W + A_T$. This is the relationship found numerically in Ref. [5]. The above simple analysis presents a simple analytic insight into the effect of the external stochasticity on $D_{\rm web}$ when the intrinsic web stochasticity is dominant over the extrinsic stochasticity.

III. DIFFUSION DOMINATED BY EXTRINSIC NOISE

Since the separatrix thickness decays exponentially on K, we have $A_W/A\propto\exp[-c/K]$ where c is a constant [9]. Thus, the diffusion rate $D^l_{\rm web}$ based upon the web stochasticity decays rapidly as K is reduced, and the extrinsic stochasticity can dominate the diffusion by scattering the phase points over the thin web layer into the neighboring tiles.

Taking the limit of small web thickness, we neglect the intrinsic stochasticity in order to simplify the analysis. The tiles are now entirely composed of regular KAM surfaces bounded by a sufficiently thin separatrix network in-between the tiles. During a complete regular rotation within the tile a phase point experiences external noise which scatters them off the constant Hamiltonian KAM curves. The scattered phase point no longer has a closed orbit, resulting in a mismatch Σ in the direction perpendicular to the KAM curves after one complete rotation. This is the basic step size across the KAM surfaces in a rotation, driven by extrinsic noise. When the extrinsic noise scatters the phase point into another neighboring tile as a result of the rotation and Σ motions, an enhanced extrinsic diffusion occurs with the step size now equal to the tile size. The average size of Σ can be estimated from a random walk diffusion

$$
\Sigma\simeq lN_0^{1/2}, \qquad (6)
$$

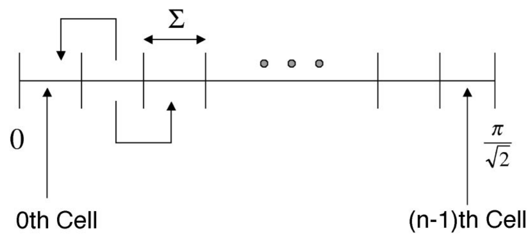

FIG. 2. Sketch of the partitioning of a tile into n cells in the $u=v$ direction.

where N_0 is an average number of mapping steps per complete rotation within the tile in the absence of extrinsic noise. For an estimate of N_0 we can use the following argument in terms of the normalized action w, which goes from zero on the separatrix to unity at the tile center. Near $w=1$, we have a regular rotation around an elliptic fixed point. Thus, the period is $N(w\rightarrow 1)=\pi/K$. N increases monotonically as w is reduced. According to Ref. [5], this rotation period approaches $N_x \simeq 2(2-\ln w)/K$ near the separatrix, where it exhibits logarithmic divergence due to hyperbolic fixed points. For a qualitative averaging between these two behaviors of N, we simply take the value of N_x far away from the separatrix $[N_x(w\rightarrow 1)]$:

$$N_0 \simeq 4/K.$$

Substituting this N_0 into Eq. (6), we obtain

$$\Sigma \simeq 2l/K^{1/2}. \tag{7}$$

Since the radial size of the tile is $\simeq \pi/\sqrt{2}$ in the $\vec{u}+\vec{v}$ direction (see Fig. 1), there can be two different diffusion regimes depending upon the magnitude of Σ relative to $\pi/\sqrt{2}$. If $\Sigma > \pi/\sqrt{2}$, the extrinsic noise can detrap the phase point out of a tile before the completion of an internal rotation. If $\Sigma < \pi/\sqrt{2}$, on the other hand, a phase point experiences many rotations before detrapping. For convenience, we define a normalized parameter for the extrinsic noise strength: $\hat{l}=(2\sqrt{2}/\pi)(l/\sqrt{K})$. In the high extrinsic noise regime characterized by $\hat{l}>1$, the qualitative evaluation of the diffusion coefficient is trivial. Since the radial step size Σ per rotation is greater than the tile size, the regular rotation motions and tile structures are destroyed by the extrinsic noise. The global diffusion in this case is basically set by the large individual random walks per mapping step, driven by extrinsic noise. Neglecting the intrinsic kick effect in the mapping equation (1), the square of the random walk size per mapping step is $(\delta r)^2=(\delta u)^2+(\delta v)^2=u_r^2+v_r^2$, yielding the diffusion coefficient D_l in the l-dominant regime as

$$D_l=\langle u_r^2+v_r^2\rangle_\eta/2=(l^2/2)\int_{-1}^{+1} d\eta\,\eta^2=l^2/3, \tag{8}$$

where η is a homogeneously distributed random number between ± 1.

In the small extrinsic noise regime, $\hat{l}<1$ (equivalently, $\Sigma<\pi/\sqrt{2}$), the diffusion process is dictated by the regular rotations and the two-dimensionally periodic tiles. Analytic derivation of the diffusion coefficient in this regime was attempted in Ref. [5]: however, we find that it requires more elaborate and proper consideration than what was reported in Ref. [5]. We partition a tile in Fig. 1 along the $u=v$ line into n discrete cells. The size of a cell is taken to be Σ. Thus, $n=\pi/\sqrt{2}\Sigma$. A phase point initially at the kth cell can scatter by the extrinsic noise either into the $(k+1)$th or the $(k-1)$th cell after one rotation. In Fig. 2 the zeroth cell corresponds to the center of the unit tile, which is on contact with a perfectly reflecting boundary. The $(n-1)$th cell represents the last one before detrapping into the neighboring tile (nth cell): Thus, the nth cell itself is an absorbing boundary.

We now evaluate the average detrapping time from a tile. The relevant quantity here is the mean number of rotations C_k before the phase point hits the absorbing boundary into the nth cell starting from the kth cell. Transition probability to the right (left) cell is denoted as $p(q)$. Let $P(T|s=k)$ be the probability for the phase point initially at the kth cell to reach the absorbing boundary after T rotational steps. The recursion equation for C_k $(1\leqslant k\leqslant n-1)$ can, then, be obtained as follows:

$$\begin{aligned}
C_k &= \sum_{T=1}^{\infty} P(T|s=k)T \\
&= \sum_{T=1}^{\infty} T[pP(T-1|s=k+1) \\
&\quad + qP(T-1|s=k-1)] \\
&= p\sum_{T=2}^{\infty} TP(T-1|s=k+1) \\
&\quad + q\sum_{T=2}^{\infty} TP(T-1|s=k-1) \\
&= p[1+C_{k+1}]+q[1+C_{k-1}] \\
&= 1+pC_{k+1}+qC_{k-1},
\end{aligned} \tag{9}$$

where $p+q=1$ and $\Sigma_{T=1}^{\infty}P(T|s=k\pm 1)=1$ have been used. Under the appropriate boundary conditions, $C_n=0$ and $C_0=1+C_1$, the general solution to Eq. (9) is obtained as follows (see the Appendix):

$$C_k=\begin{cases} \dfrac{2}{(1-q/p)^2}\left[\left(\dfrac{q}{p}\right)^{n+1}-\left(\dfrac{q}{p}\right)^{k+1}\right]+\dfrac{n-k}{p-q}, & p\neq q \\[2ex] n^2-k^2, \quad p=q=\dfrac{1}{2}. \end{cases} \tag{10}$$

Since the extrinsic noise in the present work does not prefer right from left, we use $p=q=0.5$, thus $C_k=n^2-k^2$.

For a time asymptotic behavior, we need to consider the trapping time of a newly migrated phase point starting its activity within the tile at the $(n-1)$th cell. Thus, k should be

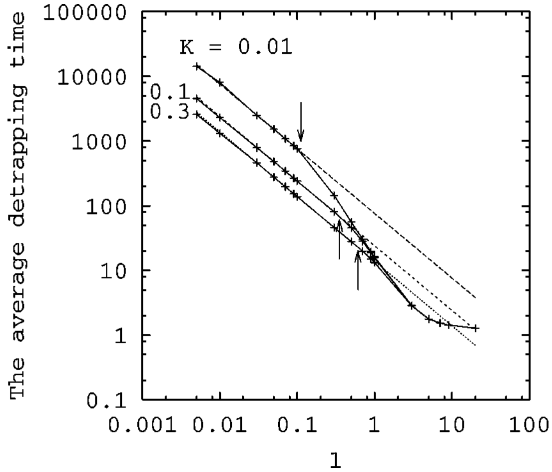

FIG. 3. The average detrapping time τ_{av} in a number of mapping steps as a function of the noise magnitude l. Numerical simulation results are marked with $+$. The dashed and dotted lines are from the fitting $2.43\pi/l\sqrt{K}$ and the arrows indicate the position $\hat{l}=1$.

$n-1$ and $C_{n-1}=2n-1\simeq 2n\simeq\sqrt{2}\pi/\Sigma$. The average trapping time τ_{av} in a tile is, therefore, given as

$$\tau_{av}=C_{n-1}N_0\simeq\frac{2^{1/2}\pi}{\Sigma}N_0\simeq 2\frac{\pi}{l}\left(\frac{2}{K}\right)^{1/2}. \tag{11}$$

Before proceeding to the evaluation of the diffusion coefficient, a numerical calculation of τ_{av} is performed in order to check the accuracy of the analytic expression (11). The result is shown in Fig. 3. The $+$ marks are numerical simulation results for $K=0.01,0.1,0.3$. The abrupt change of the numerical τ_{av} behavior around $\hat{l}=1$ (indicated by the arrows) supports the validity of the $\hat{l}$ parameter. The simple analytic prediction of Eq. (11) is shown as straight lines with a constant multiplication factor of 0.86, showing an excellent fit to the numerical result in the small extrinsic noise regime ($\hat{l}<1$). With this confidence in the averaged theory, we now estimate the radial diffusion coefficient D_K in the regime $\hat{l}<1$ using a random walk argument. The step size in this diffusion is the distance between the centers of the neighboring tiles in the $\vec{u}+\vec{v}$ direction, $\sqrt{2}\pi$, and the random walk time is τ_{av}. Thus, we have

$$D_K=\frac{(\sqrt{2}\pi)^2}{2\tau_{av}}=(\pi/2\sqrt{2})l\sqrt{K}. \tag{12}$$

Notice here that the diffusion coefficient in Eq. (12) is a function of not only l, but also K, as the numerical simulation showed. On the other hand, the crude analytic theory presented in Ref. [5] had the l dependence only ($D\propto l$) even though both results are obtained in the same regime $\hat{l}<1$ characterized by fast particle rotation within the tile compared to the extrinsic noise strength. The critical difference (and a significantly improved understanding) here is that the fast rotation of the particles within the tile makes the rotation to be the basic unit of random walk in the present work, instead of assuming the individual mapping step to be the basic unit as done in Ref. [5]. In the present work, the rotation time N, the step size Σ, and the tile-to-tile crossing probability are all functions of K; thus, the resulting τ_{av} and D_K have to be functions of K. In Ref. [5] all these physical quantities were independent of K. We find that the completely random walk behavior between every mapping step is only satisfied in the regime $\hat{l}>1$ where the tile structures are destroyed by the strong extrinsic noise, yielding $D\propto l^2$. Reference [5] did not investigate this regime, however.

The proportionality constant for D_K in Eq. (12) can be further improved by considering the directional correlation effect for the tile to tile transition, which arises from the fact that the probability for the exit direction out of a tile is correlated to the entrance direction into the tile and the rotation direction of the phase point within the tile [10,11]. This effect usually enhances the numerical coefficient somewhat with a correction coefficient of order unity, but does not alter the diffusion scaling. Thus, for a qualitative analytic study in the present work, such an elaboration is not necessary and will not be included.

IV. COMPARISON WITH NUMERICAL SIMULATION RESULT

In order to verify the validity of the analytic results, the discrete mapping equation (1) is studied numerically. Since all the phase space is interconnected by the extrinsic noise, the global diffusion coefficient is measured by breaking a single orbit into N pieces and giving each of them T mapping steps. Thus, the total length of the single orbit is NT. Numerical diffusion coefficient in $r=(u^2+v^2)^{1/2}$ is then obtained from

$$D_r=\frac{1}{2T}\frac{1}{N}\sum_{i=1}^{N}(\vec{r}_i-\vec{r}_{(i-1)})^2. \tag{13}$$

Here, T must be sufficiently large to ensure that the transient behavior dies out, and N must be large enough to provide meaningful statistics. The T and N values are chosen after comparison with a time-dependent diffusion coefficient obtained by distributing initial phase points $\vec{r}_{i,0}$ of N separate phase points uniformly in a tile and following each orbit for a sufficiently long time period T':

$$D(T')=\frac{1}{2T'}\frac{1}{N}\sum_{i=1}^{N}(\vec{r}_{i,T'}-\vec{r}_{i,0})^2. \tag{14}$$

If we can observe that the equilibrium diffusion coefficient D_r from Eq. (13) agrees with $D(T)$ from Eq. (14) for the slowest diffusion process involved in each plot (smallest l and K case), then we assume that the transient behavior has died out in the single orbit method, Eq. (13). As a result of this exercise, the values of T and N are chosen to be 1 000 000 and 5000, respectively, for all the plots in the present work unless otherwise specified. Figure 4 shows an example of the time dependent $D(T)$ from Eq. (14) for $N=5000$ and $l=0.005$. It can be seen that the transient be-

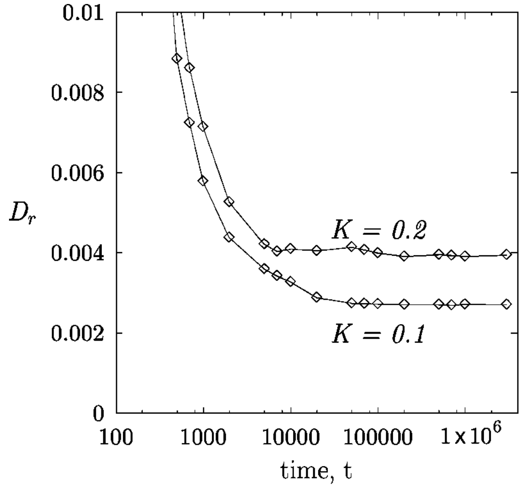

FIG. 4. Time dependence of the diffusion coefficients (radian^2/mapping step) from Eq. (14) for 5000 initial phase points distributed uniformly in the square region ($0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant \pi$). l is set to 0.005.

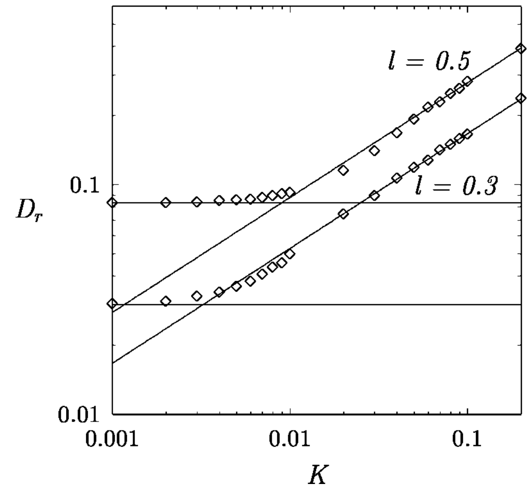

FIG. 6. Plot of the numerically obtained diffusion coefficients (radian^2/mapping step, represented by $\diamond$) versus K for two different l values (0.3 and 0.5). The horizontal lines are drawn according to Eq. (8) for $\hat{l}>1$. For $\hat{l}<1$ (the greater K region) the lines are from $1.76l\sqrt{K}$ as in Fig. 5.

havior lasts longer for smaller K, but by the time $T=10^6$ the diffusion coefficient is safely in steady state for both cases.

In Fig. 5 the numerical diffusion coefficient with $l\neq 0$ is plotted with $\diamond$ marks, with the fitting $1.76l\sqrt{K}$ shown in solid lines based upon the analytic expression, Eq. (12). Actually the fitting result $1.76l\sqrt{K}$ is somewhat greater than the prediction given by Eq. (12) ($D_K \simeq 1.11l\sqrt{K}$), not a surprising discrepancy when we consider the neglect of correlation phenomena in the transitional process between neighboring tiles as mentioned in Sec. III. In this plot the deviation of the numerical curves from the fitting lines (at high K values) are due to the onset of the stochastic web diffusion of Eq. (5). The dotted line with $+$ marks shows the global web diffu-

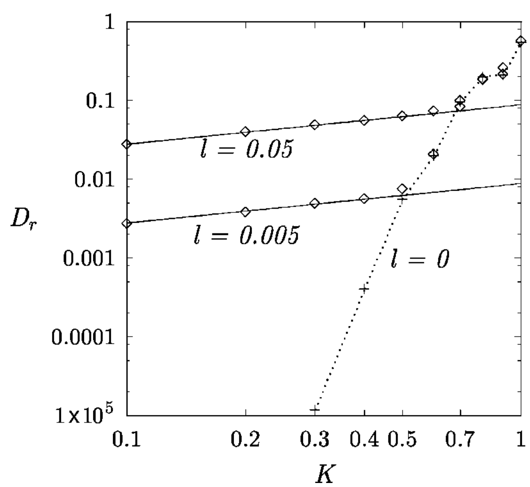

FIG. 5. Numerical diffusion coefficients (radian^2/mapping step) versus K. The numerical points marked $\diamond$ are obtained by following one orbit for a long time (5×10^9) in the presence of the extrinsic noise. The curve with $+$ marks is the web diffusion, obtained numerically without extrinsic noise by averaging over 5000 phase points uniformly distributed in the square region ($0 \leqslant u \leqslant \pi$, $0 \leqslant v \leqslant \pi$). Two solid lines are from $1.76l\sqrt{K}$.

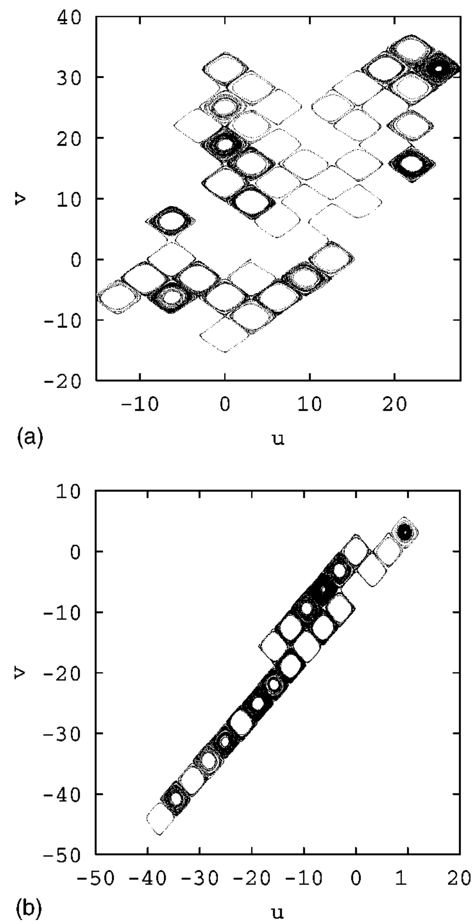

FIG. 7. One particle trajectory under the influence of extrinsic noise starting from $(u,v)=(3.0,0.0)$. $K=0.1$, $l=0.05$ and the number of iterations is 25 000. (a) $u_r\neq v_r$, (b) $u_r=v_r$.

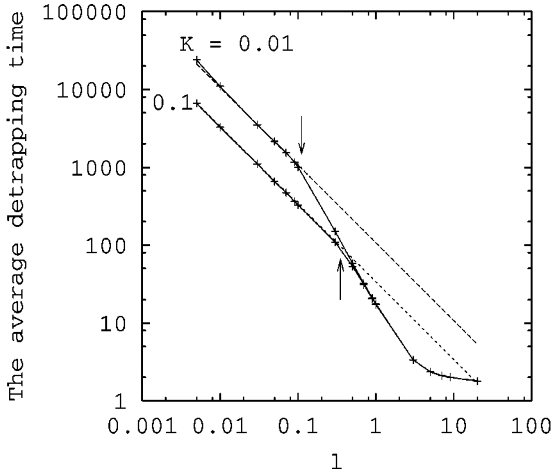

FIG. 8. The same case as for Fig. 3 except that $u_r=v_r$ is used in Eq. (1) for the numerical evaluation of τ_{av} in a number of mapping steps. The dashed and dotted lines represent $1.4\times 2.43\pi/l\sqrt{K}$.

sion without the extrinsic noise, with the initial phase points distributed uniformly over the entire tile, which verifies that the deviation from the $l\sqrt{K}$ behavior at the high K values is from the web diffusion reduced by the area ratio of web to tile. Transition into the regime $\hat{l}>1$ cannot be seen in Fig. 5, but is shown in Fig. 6. Here again the numerical results agree well with the analytic predictions, i.e., $D_K\simeq 1.11l\sqrt{K}$ of Eq. (12) and $D_l=l^2/3$ of Eq. (8).

V. COHERENCE IN THE NOISES

In some situations the random extrinsic noises u_r and v_r can originate from the same physical event and be coherent. In order to model this situation, we set $u_r=v_r$ and compare the result with the previous cases where u_r and v_r are completely uncorrelated. Figures 7(a) and 7(b) compare the tile to tile transition processes for $u_r\neq v_r$ and $u_r=v_r$. Transitions are close to isotropic for $u_r\neq v_r$ in Fig. 7(a), but are highly directional for $u_r=v_r$ in Fig. 7(b) to the $\delta u\,\delta v>0$ direction. This phenomenon can be easily understood from the condition $u_r=v_r$ that the phase points are difficult to cross the constant Hamiltonian separatrix, $v-u=$ const. Thus, the tile-to-tile transitions should be highly anisotropic, severely restricting transitions in the direction $\delta u\,\delta v<0$. Figures 8 and 9 show that τ_{av} is enhanced and D in the $r=\sqrt{u^2+v^2}$ direction is reduced by the condition $u_r=v_r$ when compared to the same cases under $u_r\neq v_r$. This phenomenon can also be understood easily from the fact that the condition $u_r=v_r$ reduces the random walk in the $\delta u\,\delta v<0$ direction, thus, the contribution to Σ is also reduced. This makes τ_{av} greater and D smaller. However, if we evaluate the diffusion in the unrestricted direction only, the diffusion coefficient is not reduced.

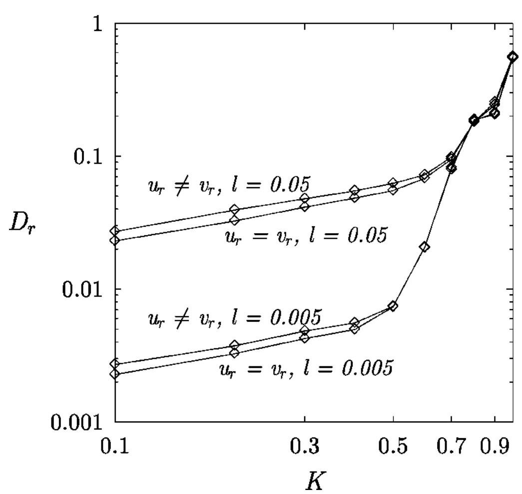

FIG. 9. Numerically obtained D_r (radian^2/mapping step). Slightly higher diffusion is obtained for $u_r\neq v_r$.

We note here that the coherent extrinsic noise $u_r=v_r$ does not change the basic scaling of the diffusion process from the incoherent noise case $u_r\neq v_r$: In both cases τ_{av} and D have the same dependencies on K and l. If we switch the phase of the coherence to $u_r=-v_r$, then the direction of the diffusion is now switched to the $\delta u\,\delta v<0$ direction. The highly directional nature of the diffusion in the coherent noise case can be an important effect in a physical situation where the two directions have different meanings.

VI. CONCLUSION AND DISCUSSIONS

In the present work, analytic understanding of the extrinsic noise effect on a nonlinear dynamical system is reported by adding random noise terms to the kicked Harper map, where the phase space is divided into infinitely periodic two-dimensional tiles. This problem has previously been studied numerically in the literature. However, the analytic understanding has been rather incomplete. It is shown in the present work that there are clean and simple ways of explaining the extrinsically driven diffusions in this physically important system.

The extrinsic noise performs three functions: First, it interconnects all the regions in the phase space. The divided spaces are no longer isolated. One significant consequence of this function is the slowing down of the stochastic web diffusion rate, when it provides the dominant global diffusion process, by moving the phase points in and out of the fast web diffusion region. This has been pointed out by previous numerical studies. Second, it makes the phase points stray off the regular constant Hamiltonian paths if the intrinsic stochasticity is not present. When the extrinsic motion becomes dominant over the regular rotation motions, the existence of the phase-space tile structures is destroyed and the diffusion becomes of the well-known random walk type ($\propto l^2$). When extrinsic motion is only a perturbation to the regular rotation motion, the diffusion closely obeys all the rules set out by the properties of the mapping including the rotations along the KAM curves in the tile and periodic two-dimensional array of tiles. Third, it forces the phase points to make transitions to the neighboring tiles, making the random walk size as large as the tile size. This gives a tremendous enhancement of the global diffusion rate over the intrinsic diffusion rate when the intrinsic perturbation is small, and yields diffusion of the type ($D\propto l\sqrt{K}$).

Depending upon the relative strength of the extrinsic noise to the intrinsic perturbation, we have defined three different diffusion regimes. The above properties combine together to give distinctive diffusion property in each of these regimes. The analytic understanding presented in the present work compares excellently with the numerical results in the literature or in the present work.

In addition we have examined the case when the two noises in each of the two dimensions are from the same physical source and are strongly correlated. We have shown that the diffusion becomes highly anisotropic, but keeps the same basic scaling laws as the case with uncorrelated noises. This problem can be important in a physical situation where the extrinsic noise is from the same physical source and the two directions have different significance.

The correlation effect over the consecutive transition steps between neighboring tiles has not been included in the present paper, which aims at qualitative results. This correlation effect can make a minor improvement on the magnitude of the diffusion coefficient when the diffusion process is dominated by the small perturbation of the regular rotation motions along the constant Hamiltonian curves.

It is interesting to note that the observations made here based upon the kicked Harper map can be connected to the passive particle (e.g., dye particles) transport in a two-dimensional periodic laminar convective flow (cellular flow) field such as the Rayleigh-Bénard convection cell. In terms of the Peclet number $P=vd/D_m$ [where v is the flow velocity in a convective cell, d is the characteristic cell size, and D_m is the local (or molecular) diffusivity originated from fluctuations or collisions] which measures the relative importance of the convective motions compared to diffusion, the global effective diffusion coefficient in the large space-time scale has been known [12–14] to be $\sim D_m P^{1/2}$. Since our noise coefficient l and perturbation parameter K can be interpreted to have similar physical functions as $\sqrt{D_m}$ and v, respectively, it can be seen readily that our formula $\sim l\sqrt{K}$, which can be translated to be valid in the large Peclet number limit, is similar to the scaling $D_m P^{1/2} = D_m\sqrt{vd/D_m} \propto \sqrt{D_m}\sqrt{v}$.

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APPENDIX: MEAN NUMBER OF ROTATIONS BETWEEN TILE TRANSITIONS

The mean number of regular rotations of a phase point between two consecutive tile transitions is given as the solution to Eq. (9),

$$C_k = 1 + pC_{k+1} + qC_{k-1}, \tag{A1}$$

which is nothing but an inhomogeneous difference equation. In the case $p=q=1/2$, the solution is simple: The above equation becomes, after a trivial rearranging,

$$-1 = \frac{1}{2}(C_{k+1} - 2C_k + C_{k-1}) \Rightarrow \frac{1}{2}\frac{d^2C_k}{dk^2}.$$

A general solution is

$$C_k = A + Bk - k^2.$$

The boundary conditions $C_n=0$ and $C_0=1+C_1$ then require $B=0$ and $A=n^2$ to yield the solution

$$C_k = n^2 - k^2 \quad \text{for} \quad p=q=1/2. \tag{A2}$$

In the case $p\neq q$ we decompose the solution to Eq. (9) into a homogeneous solution $C_k^H = pC_{k+1}^H + qC_{k-1}^H$ and the particular solution C_k^P. The homogeneous solution satisfies the homogeneous equation

$$C_k^H = pC_{k+1}^H + qC_{k-1}^H.$$

If we look for a solution in the form $C_k^H = X^k$, we have an equation for X,

$$pX^2 - X + q = 0,$$

which yields two independent solutions $X=q/p$ and $X=1$. Thus,

$$C_k^H = (q/p)^k \quad \text{or} \quad 1.$$

The particular solution is obtained from

$$C_k^P = 1 + pC_{k+1}^P + qC_{k-1}^P,$$

which can be easily converted into the differential form

$$p\frac{d^2C_k^P}{dk^2} + (p-q)\frac{dC_k^P}{dk} + 1 = 0.$$

The particular solution to this equation is

$$C_k^P = -k/(p-q).$$

After combining the homogeneous and particular solutions into

$$C_k = A(q/p)^k + B - k/(p-q)$$

and applying the boundary conditions $C_n=0$ and $C_0=1+C_1$, we obtain

$$C_k = \frac{2}{(1-q/p)^2}\left[\left(\frac{q}{p}\right)^{n+1} - \left(\frac{q}{p}\right)^{k+1}\right] + \frac{n-k}{p-q} \quad \text{for} \quad p\neq q. \tag{A3}$$

Equations (A2) and (A3) are the two components of Eq. (10).

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